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Fluctuation growth and saturation in nonlinear oscillators on the threshold of bifurcation of spontaneous symmetry breaking

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We study prebifurcation fluctuation amplification in nonlinear oscillators subject to bifurcations of spontaneous symmetry breaking which are manifest in the doubling of stable equilibrium states. Our theoretical estimates of both the linear growth and the nonlinear saturation of the fluctuations are in good agreement with our results from numerical simulations. We show that in the saturation mode, the fluctuation variance is proportional to the standard deviation of the external noise, whereas in the linear mode, the fluctuation variance is proportional to the noise variance. It is demonstrated that the phenomenon of prebifurcation noise amplification is more pronounced in the case of a slow transition through the bifurcation point. The amplification of fluctuations in this case makes it easier to form a symmetric probability of the final equilibrium states. In contrast, for a fast transition through the bifurcation point, the effect of amplification is much less pronounced. Under backward and forward passages through the bifurcation point, a loop of noise-dependent hysteresis emerges here. We find that for a fast transition of the nonlinear oscillator through the bifurcation point, the probability symmetry of the final equilibrium states is destroyed.

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I. INTRODUCTION

In nonlinear systems, pregeneration noise amplification is commonly observed at the threshold of self-sustained oscillations. This was shown for both radiophysical and optical self-sustained oscillating systems [1]. Noise amplification near the generation threshold is conditioned by the decrease of losses in the oscillator casting the real part of one of the Lyapunov indices of the system from negative to positive values. The initial state of the system loses its stability and the amplified pregeneration noise turns into an effective push for self-sustained oscillations.

Pregeneration of noise amplification is a particular case of a more general phenomenon, the prebifurcation amplification of fluctuations [2] and of weak signals due to the decrease (down to zero at the critical point) of the damping strength [3–5]. The linear theory developed in [3–5] predicts unlimited growth of fluctuations when approaching the bifurcation point.

Nonlinear analysis of prebifurcation noise amplification was performed in [6] for the case of period doubling bifurcations in the logistics map. In this case, similarly to the general one, the Lyapunov exponent changes from negative to positive. However, in general, the change of the sign of the real part Re λ of Lyapunov exponent λ now leads not to emerging generation of oscillations but to the transition of the system from the unstable state to one of the two possible stable states.

The aim of this work is to study this phenomenon of prebifurcation noise amplification in a nonlinear oscillator subject to a bifurcation of spontaneous symmetry breaking (a pitchfork bifurcation). Such a bifurcation is known to lead to two new stable states instead of one stable equilibrium state which loses its stability. The increase of the fluctuations is just due to a decrease of the linear frequency of the oscillator. As the linear frequency goes to zero, the nonlinear effects become important, and a self-consistent analysis has to be carried out.

In Sec. II, we describe the model of a nonlinear oscillator where doubling of stable equilibrium states may take place. Examples of such a system are one-dimensional cross oscillations of a rod squeezed along its axis (see Sec. II). In Secs. III–V, we present analytical and numerical estimations of the fluctuation level under fast and slow changes of a control parameter. Section VI presents results of numerical modeling. Section VII describes the phenomenon of noise-dependent hysteresis in the system under consideration. Finally in Sec. VIII, it is shown that under fast bifurcation transitions in the nonlinear oscillator, the probability symmetry of the stable final states is destroyed.

II. DYNAMIC MODEL

We start by considering oscillations in a nonlinear oscillator described by the second-order equation

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$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \frac{\partial U(x)}{\partial x} = \eta(t). \quad (1)$$

Here γ is the damping strength, $\eta(t)$ is the noise forcing the system, and $U(x)$ is the potential energy.

As is well known, the number of minima of the potential energy profile $U(x)$ in the nonlinear oscillator determines the number of equilibrium states [7]. Under the bifurcation of spontaneous symmetry breaking, a transition from a one-minimum potential to a two-minimum one takes place. This is accompanied by a doubling of the number of stable states. The potential

$$U(x) = Ax^4 + Bx^2 \quad (2)$$

corresponds to bifurcation of spontaneous symmetry breaking. For $B>0$, it has one minimum $U_{\min}=0$ (at $x=0$), and for $B<0$, there are two equal negative minima $U_{\min}(x^{\pm})=-|B^2|/4A$ located at $x^{\pm}=\pm\sqrt{|B|/2A}$ with the maximum $U_{\max}=0$ in between at $x=0$.

Assume parameter B to be dependent on time, $B=B(t)$, taking positive values $B>0$ for $t<t^*$ and negative ones for $t>t^*$. Such behavior is demonstrated, e.g., by the function

$$B(t) = -\,\omega_0^2 \arctan g\beta(t-t^*). \quad (3)$$

The coefficient β characterizes here the speed of the change of the control parameter.

For a physical prototype of the system described by potential (2), we can take one-dimensional cross oscillations (along the axis x) of a flat rod (ruler) under the influence of a squeezing force along its axis that grows in time. At a critical squeeze when B turns to zero, the rod undergoes a bifurcation of spontaneous symmetry breaking and takes a curved shape corresponding to one of the two stable states x^+ or x^- [8].

The goal of this paper is to determine the variance $\langle[x(t)-\bar{x}]^2\rangle\equiv\sigma_x^2$ of the response of $x(t)$ to the fluctuation force $\eta(t)$ ($\bar{x}$ is the mean value equal to the stationary value) and in that way we describe both the growth and nonlinear saturation of the prebifurcation noise amplification shortly before the bifurcation.

We will assume that the fluctuation force $\eta(t)$ is a stationary random process with the autocorrelation function

$$\langle\eta(t')\eta(t'')\rangle = \sigma_\eta^2\Psi_\eta(t'-t''), \quad (4)$$

where σ_η^2 is the variance and $\Psi_\eta(\tau)$ is the autocorrelation function. We define the correlation time τ_η as the time when the autocorrelation function goes down to the value 0.5. Below we will restrict attention to short-time correlated processes, i.e., τ_η is small compared to the period of the free oscillations $2\pi/\omega_0$ of the unperturbed oscillations, so that $\omega_0\tau_\eta \ll 2\pi$. Hence, the noise is similar to white noise.

III. ESTIMATION OF FLUCTUATIONS UNDER SLOW CHANGE OF SYSTEM PARAMETERS: WKB APPROXIMATION

Under a low amplitude of the oscillations and for $B(t)>0$, Eq. (1) can be linearized

$$\ddot{x} + 2\gamma\dot{x} + \omega^2(t)x = \eta(t), \quad (5)$$

where $\omega(t)=\sqrt{B(t)}$. For a slow enough decrease of $B(t)$, i.e., when $\beta\ll\omega_0$, $x(t)$ is fairly well described by the Wentzel-Kramers-Brillouin (WKB) approximation:

$$x(t) = \frac{A_x}{\sqrt{\omega(t)}}e^{-\gamma t}\cos[\Phi(t)+\varphi_0], \quad (6)$$

where

$$\Phi(t) = \int_0^t \omega(t')dt'.$$

It is easy to apply the same approximation to the Green's function of Eq. (5):

$$g(t,t') = \begin{cases} 0 & \text{at } t<t', \\ \dfrac{e^{-\gamma(t-t')}}{\sqrt{\omega(t)\omega(t')}}\sin\Phi_g(t,t') & \text{at } t>t', \end{cases} \quad (7)$$

where

$$\Phi_g(t,t') = \int_{t'}^t \omega(t'')dt''.$$

Using the Green's function (7), the solution of the heterogeneous linear equation (5) can be written as

$$x(t) = \int_{-\infty}^t g(t,t')\eta(t')dt', \quad (8)$$

and the variance $\sigma_x^2=\langle x^2(t)\rangle$ as

$$\sigma_x^2 = \int_{-\infty}^t dt'\int_{-\infty}^t dt'' g(t,t')g(t,t'')\langle\eta(t')\eta(t'')\rangle. \quad (9)$$

For short-time correlated noise (4) and the Green's function (7), this expression takes the form

$$\sigma_x^2 = \tau_\eta\sigma_\eta^2\int_{-\infty}^t g^2(t,t')dt' = \tau_\eta\sigma_\eta^2\frac{1}{\omega(t)}\int_{-\infty}^t \frac{e^{-\gamma(t-t')}}{\omega(t')}\sin^2\Phi_g(t,t')dt'. \quad (10)$$

Substituting for $\sin^2\Phi_g(t,t')$ its mean value 1/2, we get the estimate

$$\sigma_x^2(t) = \sigma_\eta^2\frac{\tau_\eta}{2\gamma\omega(t)\Omega(t)}, \quad (11)$$

where $1/\Omega(t)$ denotes the integral

$$\frac{1}{\Omega(t)} = \gamma\int_{-\infty}^t \frac{e^{-\gamma(t-t')}}{\omega(t')}dt', \quad (12)$$

representing the value $1/\omega(t')$ time averaged with the weight $\gamma\exp[-\gamma(t-t')]$.

It is worth mentioning that for a constant frequency $\omega=\omega_0$, the expression (11) turns into the well-known expression for steady fluctuations of the linear oscillator under

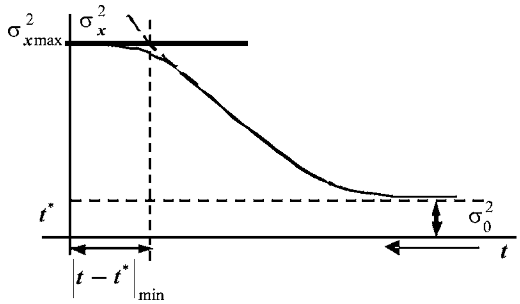

FIG. 1. Nonlinear estimate of the fluctuation variance.

short-time correlated fluctuation forcing $\eta(t)$ [1]

$$\sigma_0^2 = \sigma_\eta^2 \frac{\tau_\eta}{2\gamma\omega_0^2}. \tag{13}$$

When approaching the bifurcation point $t=t^*$, where $B(t)$ turns to zero, the linear estimate (11) describes a growing fluctuation variance reflecting the prebifurcation amplification of the noise. The solid line in Fig. 1 represents this fluctuation growth. The values $1/\omega(t)$ and $1/\Omega(t)$ from Eq. (11) go to infinity for $t \to t^*$. The linear estimate (11) follows the same path, as is shown in Fig. 1 by the dash-dotted line. Obviously, the linear estimate (11) based on the WKB approximation is not applicable in the vicinity of the bifurcation point. However, we will demonstrate in the next section that for $t \to t^*$ it is possible to obtain a nonlinear estimate for the variance σ_x^2 and will show that it takes finite values even at the bifurcation point.

A peculiarity of the bifurcation scenario consists in the fact that while approaching the point of bifurcation, the real parts of the Lyapunov exponents $\gamma = \mathrm{Re}\,\lambda$ do not go down to zero as for the Landau-Hopf bifurcation and the period doubling bifurcation. That is why the fluctuation increase here is not related to loss decrease. Instead, we have a transformation of a pair of conjugate complex exponents $\lambda_{1,2} = -\gamma \pm i\omega$ into a pair of real exponents, one of which is positive. As a result, the prebifurcation noise amplification is now connected with the decrease of the frequency $\omega = \mathrm{Im}\,\lambda$, because for $\omega \to 0$, the amplitude of the free oscillations grows as $1/\sqrt{\omega(t)}$.

IV. NONLINEAR ESTIMATIONS OF FLUCTUATION INTENSITY IN THE VICINITY OF THE BIFURCATION POINT

As discussed above, expression (11) based on linear theory loses its validity in the vicinity of the bifurcation point. This occurs due to nonlinear effects arising under infinite growth of the fluctuations. Nonlinear effects can be neglected as long as the fourth-power term in (2) is small compared to the second-order term:

$$A\langle x^4 \rangle \ll |B(t)| \langle x^2 \rangle. \tag{14}$$

Supposing the fluctuations η and x are Gaussian, for estimations we assume

$$\langle x^4 \rangle = 3\langle x^2 \rangle^2 = 3\sigma_x^4$$

and then rewrite inequality (14) as

$$|B(t)| \gg 3A\sigma_x^2 = \frac{3A\sigma_\eta^2 \tau_\eta}{2\gamma\omega(t)\Omega(t)}. \tag{15}$$

If we also assume that the characteristic oscillation attenuation time $1/\gamma$ is less than the characteristic change time $1/\beta$ of the frequency $\omega(t)$, i.e., $\gamma > \beta$, then from Eq. (12), we get

$$\frac{1}{\Omega(t)} \approx \frac{1}{\omega(t)}. \tag{16}$$

Then Eq. (15) takes the form

$$B^2(t) = \omega^4(t) \gg \frac{3A\sigma_\eta^2 \tau_\eta}{2\gamma}. \tag{17}$$

From (17), we infer the estimate

$$B_{\min} \sim \left(\frac{3A\sigma_\eta^2 \tau_\eta}{2\gamma} \right)^{1/2} \tag{18}$$

for the permitted distance to the point of bifurcation, and from (11) together with (16), we estimate

$$\sigma_{x\,\max}^2 \sim \sigma_\eta \sqrt{\frac{\tau_\eta}{6\gamma A}} \tag{19}$$

for the maximum fluctuation intensity. Therefore, for $B < B_{\min}$, the linear effects of fluctuation growth are replaced by the nonlinear saturation to be reached at the fluctuation intensity $\sigma_{x\,\max}^2$. The horizontal line in Fig. 1 denotes the value $\sigma_{x\,\max}^2$. A similar saturation of the fluctuation intensity is found for period doubling bifurcations as well [6]. A general approach for estimating the saturation level is discussed in [9].

It is useful to introduce the prebifurcation noise amplification factor K as the ratio of $\sigma_{x\,\max}^2$ to the fluctuation intensity (13) at $\omega = \omega_0$:

$$K = \frac{\sigma_{\max}^2}{\sigma_0^2} = \frac{1}{\sigma_\eta} \left(\frac{2\gamma}{3A\tau_\eta} \right)^{1/2} \omega_0^2. \tag{20}$$

This value indicates how many times the fluctuation intensity in the saturation zone is greater than the stationary fluctuation intensity of the oscillator.

The phenomenon of prebifurcation increase of fluctuation intensity is accompanied by another phenomenon—prebifurcation correlation time increase. The latter also experiences saturation in the vicinity of the bifurcation point. In fact, one can speak about a prebifurcation rise and subsequent saturation of the correlation time. This phenomenon is found for period doubling bifurcations [10]. It is noteworthy that the phenomenon of prebifurcation correlation time increase has a general nature. It is also observed in a nonlinear oscillator subject to bifurcations of spontaneous symmetry breaking and described in the present paper. This approach looks to be a good prospect for analysis of different bistable systems [11] and nonlinear geophysical systems in which a period doubling bifurcation and a bifurcation of spontaneous

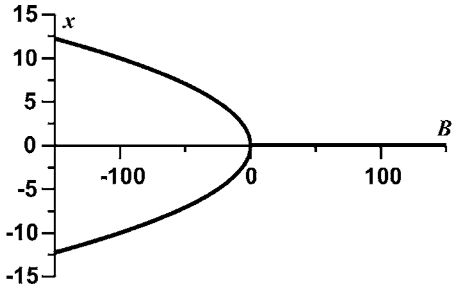

FIG. 2. Bifurcation diagram of the nonlinear oscillator [Eq. (1)].

symmetry breaking take place. Such a system is, for example, the Kuroshio current system [12].

V. FLUCTUATIONS UNDER A VERY FAST CHANGE OF THE OSCILLATOR PARAMETERS

Above we considered fluctuations for rather slow changes ($\beta < \gamma$) of the oscillator parameters. For a fast passage through the bifurcation point, that is, for $\beta > \gamma$, one may expect some decrease of $\sigma^2_{\max}$ in comparison to the case $\beta < \gamma$.

The trend to a smaller $\sigma^2_{\max}$ with growing speed β of the transition through the bifurcation point can be illustrated by the extreme case $\beta \to \infty$, when for $t < t^*$, $B(t)$ is constant and equals ω_0^2, while for $t > t^*$, it is constant as well but equals $-\omega_0^2$. In this case the fluctuation intensity at $t < t^*$ is constant and equals σ_0^2, so that the factor of the fluctuation amplification yields $K=1$.

Naturally, for $t > t^*$, we will observe an exponential growth of the fluctuations due to the stability loss of the equilibrium state $x=0$, but this will have no effect on the fluctuations for $t < t^*$.

VI. NUMERICAL SIMULATIONS

The nonlinear oscillator described above was numerically tested for $A=0.5$, and values of B ranging from $B_0=100$ to $B_f=10^{-8}$, that is, almost reaching zero. Changes of B were slow enough to be in a quasistationary mode. The damping index γ was taken equal to 0.1. So, to satisfy the quasistationary requirement, we needed the characteristic change time of B to be small compared to 0.1.

The specified range of parameter B allows us to determine the fluctuation variance both in the immediate vicinity of the bifurcation point $B_c=0$ and far away from it (let us recall that far from the bifurcation point, each value of B corresponds to an oscillation frequency $\omega_0=\sqrt{B}$). For $B>0$, when the system has only one stable solution (see bifurcation diagram in Fig. 2), this stable point $x(0)=0$ was taken as the initial value, and the initial value of the derivative $x'(0)$ was taken equal to 0.

A random number generator produced normally distributed values of $\eta(t)$ with zero mean, $\langle \eta(t) \rangle = 0$, and a mean square deviation σ_η ranging from 10^{-6} to 10^{-1}. The autocorrelation function of the random process has a characteristic time τ_η in the interval from 10^{-2} to 10^{-3}, which is small compared to $2\pi/\omega_0$. In this case, the results do not depend on the shape of the autocorrelation function Ψ_η; as it must be for processes close to white noise. The use of other random number generators producing, for example, uniformly distributed values of $\eta(t)$ gave results qualitatively similar to those obtained for normally distributed noise. The numerical solution for (1) was obtained using a fourth-order Runge-Kutta method with a fixed step.

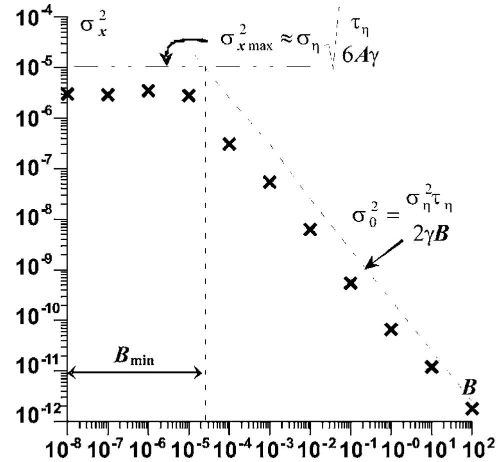

FIG. 3. Prebifurcation noise amplification for a bifurcation of spontaneous symmetry breaking. Fluctuation variance σ_x^2 versus parameter B in the quasistationary mode ($\sigma_\eta^2=10^{-8}$, $\tau_\eta=3.5\times10^{-3}$, $\gamma=0.1$, $A=0.5$).

Results of the numerical simulations are presented in Fig. 3, where the crosses show the dependence of the fluctuation variance σ_x^2 on B. We used a slow change of the parameter B (quasistationary mode) and fluctuation variance $\sigma_\eta^2=10^{-8}$. The dash-dotted line corresponds to the fluctuation variance σ_0^2 estimated in the linear mode (13) describing fluctuation growth in the direction toward the bifurcation threshold. The figure shows that the numerical results correspond well to the linear estimate for $B > B_{\min}$, which is in line with (18). With a closer approach to the bifurcation point $B=0$, the fluctuation variance reaches saturation at $\sigma^2_{\max}$, marked by a horizontal dashed line obtained from the estimate (19).

For the considered quasistationary mode, at noise variance $\sigma_\eta^2=10^{-8}$, the fluctuation amplification factor $K_{\max}$ (17) was 1.55×10^6. This value agrees in magnitude with the theoretical estimate $K_{\max}=6.17\times10^6$.

Figure 4 presents our results on the numerical simulations where we show the main property of the saturation regime: the dependence of the fluctuation variance $\sigma^2_{x\,\max}$ on the mean square of the noise forcing σ_η is linear. The dash-dotted line corresponds to the maximum fluctuation variance $\sigma^2_{x\,\max}$ (in nonlinear mode on the saturation level) obtained by formula [Eq. (19)]. The data of numerical simulations are marked by crosses. The dashed line corresponds to the result of approximation by power dependence $\sigma^2_{x\,\max}=0.192\sigma_\eta^{1.2}$. It should be noted that in these numerical simulations the phenomenon of correlation time increase appears [10]. This make difficulties and a long time for numerical simulations; therefore in the case of weak noise ($\sigma_\eta=10^{-7}-10^{-5}$) the values of maximum fluctuation variance are less than expected.

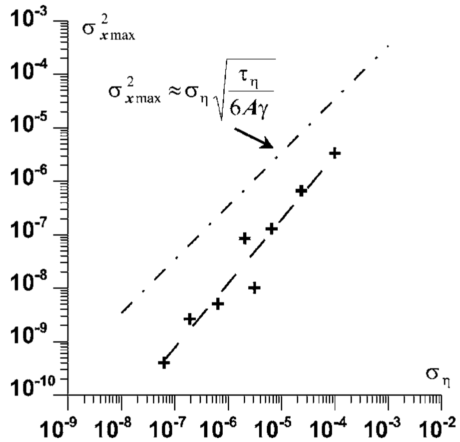

FIG. 4. Dependence of the maximum fluctuation variance $\sigma^2_{x\max}$ in the saturation regime on the mean square of the noise forcing σ_η ($\tau_\eta=3.5\times10^{-3}$, $\gamma=0.1$, $A=0.5$).

Our qualitative estimate [Eq. (19)] is in satisfactory agreement with the numerical results. The estimate [Eq. (19)] turns out to be only 40% larger as compared to numerical data.

The described results are in qualitative agreement with the data obtained earlier for period doubling bifurcations [6]: in both cases, the fluctuation mean square σ_x^2 is proportional to the mean square noise deviation σ_η, whereas the amplification factor $K_{\max}$ is inversely proportional to σ_η.

VII. NOISE-DEPENDENT HYSTERESIS IN THE VICINITY OF THE BIFURCATION POINT

Next we study the phenomenon of noise-dependent hysteresis in our system. Such a hysteresis occurs when after passing through the bifurcation point, the system for a considerable time remains on the unstable branch and only after some time makes a rather quick transition to another stable state [13,14]. The greater the speed of the control parameter change, the more distinct is the hysteretic phenomenon.

In a quasistationary mode, when the parameter B changes slowly, the bifurcation in the system occurs at the critical value $B=B^*=0$. Under a fast change of B, bifurcation of stable state doubling happens only some time after the critical value $B^*=0$ is passed, the delay "time" being dependent on the speed of change of the parameter β.

Figure 5 presents results of numerical simulations of the bifurcation transition in the nonlinear oscillator when B varies via Eq. (3). For illustrative purposes, Fig. 5 combines the bifurcation diagram of the model that is stable for constant B, but shows a hysteresis of $x(B)$ for varying B. With the change of B, the system, after passing the value $B=B^*$, still resides for some time in the vicinity of the unstable branch (this time depends considerably on speed β), and only after that switches to one of the two stable states of the equilibrium. At a high transition speed $\beta=300$ (plot 3), the delay time Δt_3 is considerably greater than the delay time Δt_2 at $\beta=0.3$ (curve 2) and the delay time Δt_1 under a very slow transition with the speed $\beta=0.03$ (plot 1). Under both forward and backward transitions through the bifurcation point, the system is known to slow down in the vicinity of the former stable points. This delay phenomenon causes the emergence of a hysteretic loop. Note that under the forward transition through the bifurcation point, the system is more sensitive to noise than under the backward transition. Figure 6 illustrates the phenomenon of hysteresis for the nonlinear oscillator at $\beta=3$ (forward sweep) and -3 (backward sweep). As is clearly seen from this figure, the hysteretic loop diminishes with noise. This effect can be used to measure weak noise in nonlinear systems as suggested earlier for systems with period doubling bifurcations [14].

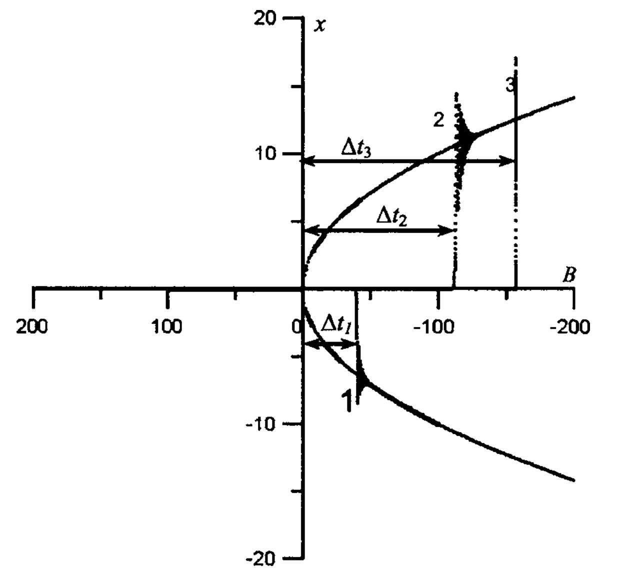

FIG. 5. Bifurcation diagram and solutions for the differential equation (1): slow transition through the bifurcation point, $\beta=0.03$ (plot 1); fast transition, $\beta=0.3$ (plot 2); very fast transition, $\beta=300$ (plot 3).

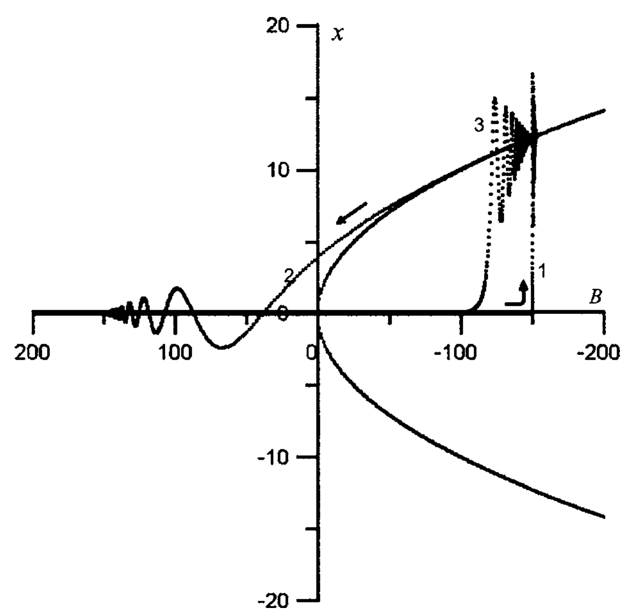

FIG. 6. Forward and backward transition through the bifurcation point at speeds $\beta=3$ (plot 1) and -3 (plot 2); the impact of noise $\sigma^2_\eta=1.87\times10^{-7}$ at $\beta=3$ (plot 3).

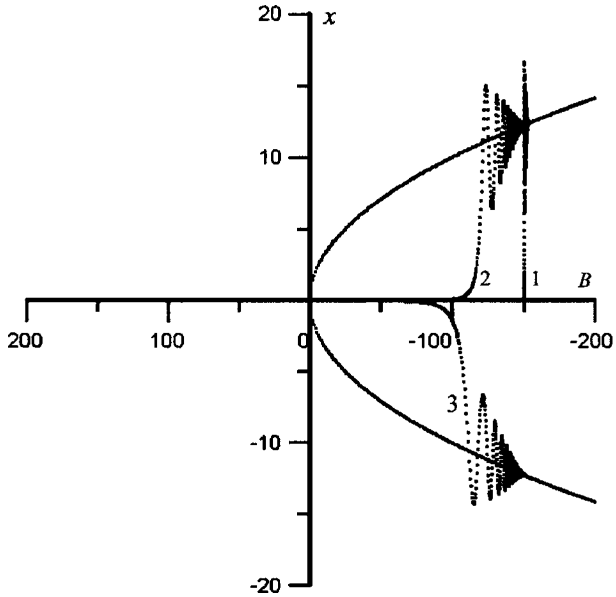

FIG. 7. Probability symmetry breaking in the nonlinear oscillator. Bifurcation diagram and solution for Eq. (1) at $\beta=3$ (plot 1); solutions in the presence of noise with variance $\sigma_\eta^2=1.87\times10^{-7}$ (plots 2 and 3).

VIII. PROBABILITY SYMMETRY BREAKING IN NONLINEAR OSCILLATOR UNDER BIFURCATION OF STABLE STATE DOUBLING

As is shown in [15,16] for period doubling bifurcations, dynamic bifurcations under low noise are characterized by a probability symmetry breaking. Under considerable noise, the probabilities of the transitions into two equitable final states are the same, each equaling 50%, while in the absence of noise, the final state of the system is fully predictable and depends only on the initial conditions and the speed of change of the control parameter. The phenomenon of probability symmetry breaking is also observed in bifurcations of spontaneous symmetry breaking. In the absence of noise, the system transits with 100% probability into one of the two possible final states determined by the speed of the transition and the initial conditions. Under the impact of noise, the probabilities of transition of a nonlinear oscillator with varying parameters into either of the two final states tend to become equal.

Figure 7 illustrates the phenomenon of probability symmetry breaking in the nonlinear oscillator. In the absence of noise, for the initial value $x_0=0$ and the initial derivative value $x'(0)=1$, for the speed $\beta=3$, the dependence of x on B is shown in plot 1. Under the impact of noise, the system may transit into either the "upper" state (plot 2), or the "lower" state (plot 3), and it will happen considerably earlier than in the absence of noise (for numerical tests, we took the noise variance $\sigma_\eta^2=1.87\times10^{-6}$). Figure 8 shows the dependence of final state probabilities on the initial values x_0. As seen in this figure, at noise variance $\sigma_\eta^2=1.87\times10^{-7}$, the limits of the final state attraction zones are smeared by noise. In contrast to discrete maps [16], the pattern of the final state attraction zones for a nonlinear oscillator depends not only on initial values and the bifurcation transition speed, but on the initial value of the derivative as well. In the case of low speeds, the attraction zones are split up, resulting in even greater system sensitivity to noise.

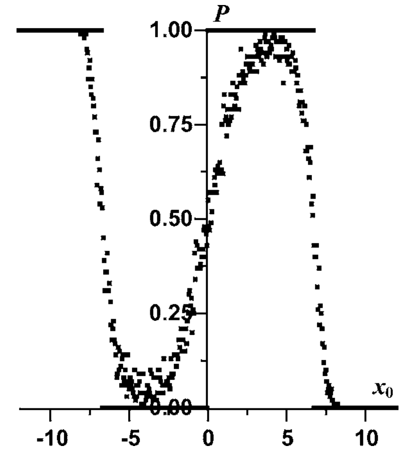

FIG. 8. Final state attraction zones (solid lines); smearing of attraction zone limits by noise with variance $\sigma_\eta^2=1.87\times10^{-7}$ at $\beta=3$ (dots).

When the transition through a bifurcation point slows down, the prebifurcation noise amplification pushes the final state probabilities to become equal. Meanwhile, under fast bifurcation transitions, the fluctuations in the vicinity of the bifurcation point decrease and the predictability of the final state rises.

IX. CONCLUSION

This work investigates fluctuations in a nonlinear oscillator subject to bifurcations of spontaneous symmetry breaking. Analytical estimates of prebifurcation noise amplification are obtained for both linear and nonlinear approximations. It is shown that the variance of the forced fluctuations σ_x^2 in the saturation mode (near the bifurcation point) is proportional to the mean square of the noise forcing σ_η, $\sigma_x^2 \propto \sigma_\eta$, whereas in the linear mode (far from the bifurcation threshold), it is proportional to the noise variance σ_η^2, $\sigma_x^2 \propto \sigma_\eta^2$. Analytical estimates are in good agreement with numerical results.

The prebifurcation fluctuation amplification is shown to facilitate the establishment of probability symmetry of the final equilibrium states. Under a slow change of the oscillator parameter, the impact of weak noise results in equalizing the probabilities of the two possible final states. Under a fast bifurcation transition, the effect of prebifurcation noise amplification weakens, the system becomes less sensitive to noise, and the final states become more predictable (probability symmetry breaking).

Finally, it is demonstrated that the bifurcation of spontaneous symmetry breaking in a nonlinear oscillator with varying control parameter is accompanied by a delay phenomenon, whose parameters strongly depend on the level of noise, a noise-dependent hysteresis.

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